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Quantal Response Analysis in the Absence of a Zone of Mixed Results Using Data Augmentation

by David W. Webb

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14. ABSTRACT

Quantal response analysis is used to estimate the probability of a dichotomous response, e.g., complete penetration, as a function of a stimulus variable. Using maximum likelihood and the DiDonato-Jarnagin algorithm, estimates of the normal distribution parameters that underlie threshold stimulus levels are obtainable if a zone of mixed results is observed in the test data and if the average success-producing stimulus exceeds the average failure-producing stimulus. In the absence of a zone of mixed results, a method is proposed that utilizes data augmentation to estimate some parameter of interest, e.g., the probability of success at a specific stimulus level. This method generates artificial copies of the original data set of stimuli and responses and then adds a random noise component to each of the stimuli. The perturbed and original data are combined into an augmented data set until a zone of mixed results is obtained and the usual analysis can proceed. Confidence statements are attainable by repeating this process to yield an empirical distribution of the parameter of interest.

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quantal response, augmentation, estimation, DiDonato-Jarnagin algorithm, zone of mixed results

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1. Introduction

Quantal response analysis is the methodology used to estimate the probability of a successful dichotomous response as a function of some stimulus variable. In a current research program being investigated at the U.S. Army Research Laboratory, a projectile is fired at a sample of developmental armor to determine if the target is able to defeat the round, i.e., if the round did not perforate the back side of the target. Projectile vs. target applications of quantal response analysis have traditionally utilized impact velocity as the stimulus variable. Realizing that other factors play a significant role in the interaction of a projectile and target, this current program considers a set of quantitative characteristics of the projectile measured from x-ray images prior to the moment of impact. These characteristics serve as input to a computer model that predicts how much penetration would have occurred if that same round was fired into semi-infinite rolled homogeneous armor (RHA). This predicted RHA penetration is an indicator of projectile lethality and therefore serves as the stimulus variable used in the quantal response analysis for this research program.

Because quantal response analysis requires that the probability of success increases with an increase in the stimulus variable, one must be careful in defining a successful outcome. Although the purpose of the study is to develop an effective armor against this particular threat, a successful outcome occurs if the projectile completely penetrates the armor—*not* if the armor defeats the round.

The underlying mathematical model of quantal response analysis assumes that each experimental unit has an associated threshold level of the stimulus variable for which an exceeded exposure will result in a successful response. In particular, each given armor sample has a corresponding RHA penetration threshold which is unknown and always will be unknown due to the destructive nature of the experiment. If the presented threat has an RHA penetration capability exceeding this threshold, then the armor will be penetrated; otherwise, the armor will not be penetrated.

Additionally, the underlying model assumes that the population of threshold values is normally distributed. The objective of quantal response analysis is to estimate the parameters (mean and standard deviation) of this normal distribution. From a probabilistic perspective, the success probability associated with a given RHA penetration is the probability that a randomly selected armor's threshold RHA penetration will be less than the given value. This probability is given by the cumulative distribution function (CDF) that results from these normal parameter estimates. That is, if r is a specific RHA penetration, then the probability of penetration into the developmental armor is given as $P(r) = \Phi\left(\frac{r-\hat{\mu}}{\hat{\sigma}}\right)$, where $\hat{\mu}$ is the estimate of the mean, $\hat{\sigma}$ is the estimate of the standard deviation, and $\Phi(\ )$ is the standard normal CDF, a function that is tabulated in most elementary statistics texts.

Estimates for the mean, μ, and the standard deviation, σ, are obtained using maximum likelihood and the DiDonato-Jarnagin algorithm.[1] For this algorithm to converge to a solution, the following conditions must be met:

1. The set of responses must include both successes and failures.
2. The average of the success-producing stimuli must exceed the average of the failure-producing stimuli.
3. The maximum failure-producing stimulus must exceed the smallest success-producing stimulus.

Figure 1 graphically shows what happens when the convergence conditions are not met, and when all of the conditions are satisfied in a quantal response study. For each case, a successful outcome is plotted as a "1" on the vertical axis; failures are plotted as "0." The case in figure 1a occurs when all outcomes are successful, while the case in figure 1b occurs when they are all failures; both of these cases are a violation of condition 1. Figure 1c shows an example of when condition 2 is not met, i.e., the average of the success-producing stimuli is less than the average of the failure-producing stimuli. The case in figure 1d is a violation of condition 3, whereby the minimum of the success-producing stimuli exceeds the maximum of the failure-producing simuli. An example for which all three convergent conditions are satisfied is shown in figure 1e.

When condition 3 is met, the region between the smallest success-producing stimulus and the maximum failure-producing stimulus is referred to as the zone of mixed results (ZMR). If no ZMR exists, an alternate method is required to obtain estimates for the parameters μ and σ.

The method advocated in this report is an example of data augmentation, a statistical technique in which artificial and observed data are combined to facilitate the analysis. Spurred by the advent of high-speed computing, augmentation has gained favor among many in the statistical community and is the basis for several recently developed methods including Markov chain Monte Carlo,[2] the Expectation-Maximization Algorithm,[3] and problems involving missing data.[4]

---

[1] DiDonato, A. R.; Jarnagin, M. P. Maximum Likelihood Estimation in Quantal Response Experiments. *SIAM J. Appl. Math* **1974,** *26* (2), 447–454.

[2] Gamerman, D. *Markov Chain Monte Carlo;* Chapman & Hall: London, 1997.

[3] Dempster, A.; Nan Laird, N.; Rubin, D. Maximum Likelihood from Incomplete Data via the EM Algorithm. *J. of the Royal Statistical Society* **1977**, *39* (1), 1–38.

[4] Kirk, R. E. *Experimental Design*; 2nd ed.; Brooks/Cole: Monterey, CA, 1982.

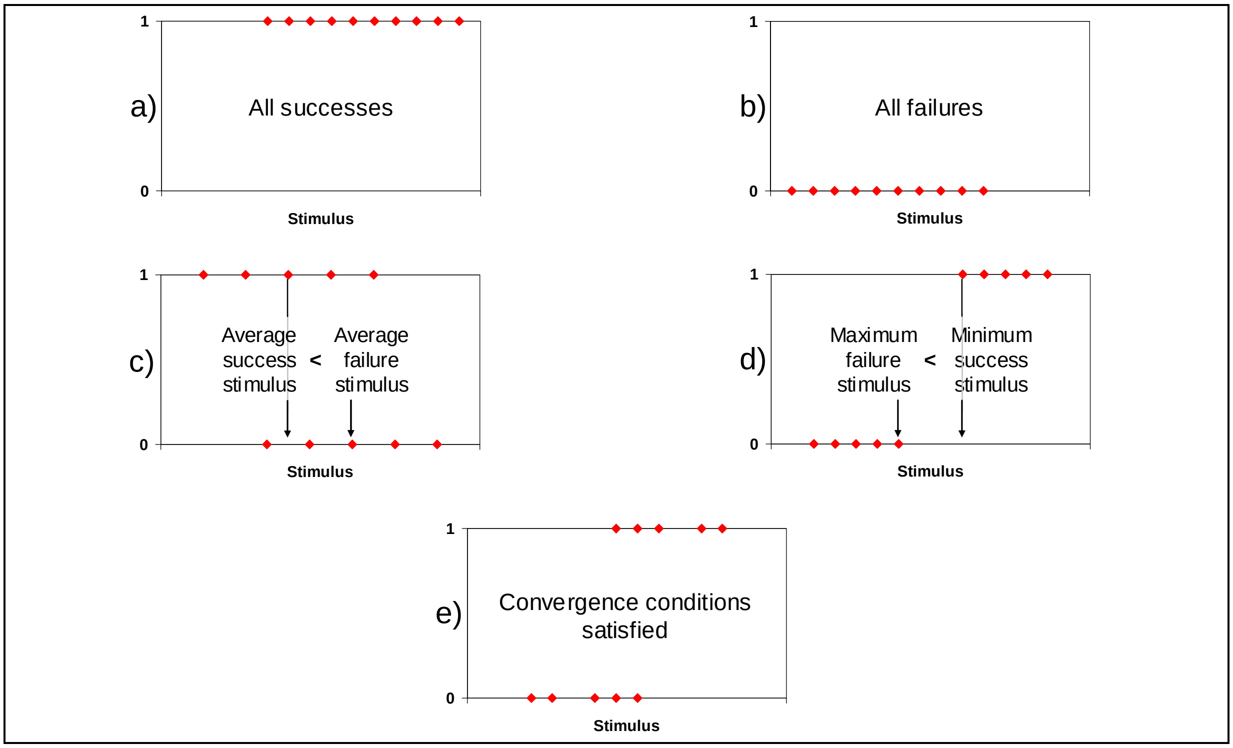

Figure 1. Examples of the five possible cases in a quantal response study.

2. Quantal Response Analysis When the ZMR Does Not Exist (Condition 3 Failure)

When a ZMR is absent, data augmentation as a method for estimating the mean and standard deviation of the threshold stimulus levels is proposed. This approach starts by generating a copy of the original data set and then adding a random noise component to each of the stimuli. The random noise components are independently drawn from a normal distribution with zero mean and a standard deviation equal to the precision with which the stimuli can be measured. This artificial data is combined with the original data to produce an augmented data set that is twice as large as the original data set and has the same ratio of successes to failures. If the augmented data contains a ZMR, then the quantal response analysis proceeds in the customary manner by estimating the normal parameters with the DiDonato-Jarnagin algorithm. However, if there is still no ZMR, then the augmentation process is continued until a ZMR is obtained.*

* If the difference between the minimum success stimulus and the maximum failure stimulus (Δ) is large relative to the standard deviation (σ^*) used to generate the noise components, repeated augmentation may not necessarily yield a ZMR. As a general rule of thumb, it is recommended that $\sigma^* > \Delta/5$ for augmentation to work efficiently.

A point estimate for either the probability of success at a given stimulus level or a particular percentile is easily calculated from the estimated μ and σ. However, since this point estimate is based on one or more random artificial data sets, it will vary from analysis to analysis (i.e., separate analyses of the same original data will produce different solutions). This may not (and should not) be acceptable to most researchers. Therefore, it is recommended that confidence intervals (or bounds) be reported in lieu of point estimates. Confidence statements are made by repeating the augmentation analysis many times to produce an empirical distribution of point estimates. Quantiles from the empirical distribution form the appropriate confidence limits.

3. Example of Augmented Quantal Response Analysis When the ZMR Does Not Exist

Consider a quantal response study in which 10 projectiles are fired at samples of developmental armor to see if they result in a complete penetration. Based on the threat characteristics immediately prior to impact, a postshot computer analysis determines how much penetration would have been achieved into semi-infinite RHA. These predicted RHA penetrations serve as the stimulus variable, while nonpenetration into the developmental armor serves as the dichotomous response. Eight of the threats failed to penetrate the developmental armor; their RHA predictions were 1026.1, 1027.2, 1035.4, 1051.8, 1052.7, 1060.0, 1073.8, and 1078.8.* Two of the threats successfully penetrated the armor with corresponding RHA predictions of 1081.6 and 1084.3. Figure 2 shows these outcomes graphically. Note that the maximum RHA prediction from a nonpenetrating round is 1078.8 and the minimum RHA prediction is 1081.6. Therefore, a ZMR does not exist, and an analysis using augmented data is warranted.

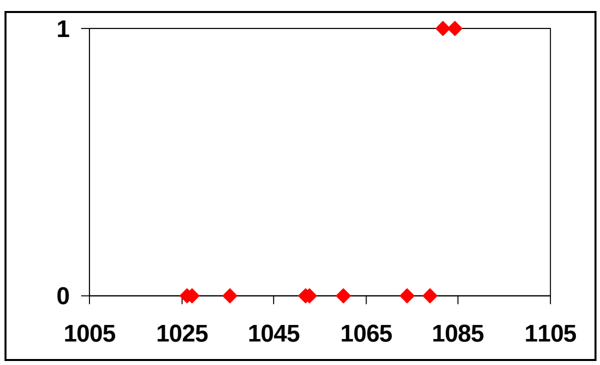

Figure 2. Data set from quantal response study with no zone of mixed results.

* Due to the sensitive nature of the actual study, the RHA penetrations reported here have been altered and the units are not specified.

The standard deviation used in generating all noise components for the artificial stimuli is 7, since this is the prediction error for the RHA penetration model. The first artificial value, drawn from a normal distribution having mean 1026.1, is a failure with RHA penetration 1024.8; the second artificial value, drawn from a normal distribution having mean 1027.2, is a failure with RHA penetration 1032.3; …; the eighth artificial value, drawn from a normal distribution having mean 1078.8, is a failure with RHA penetration 1079.2. The ninth artificial value, drawn from a normal distribution having mean 1081.6, is a success with RHA penetration 1078.5; the tenth artificial value, drawn from a normal distribution having mean 1084.3, is a success with RHA penetration 1080.9. Combining the original data and the artificial data yields an augmented set of data, which is shown in figure 3. Now the maximum failure occurs at 1081.3 and the minimum success at 1078.5, so that a ZMR exists.

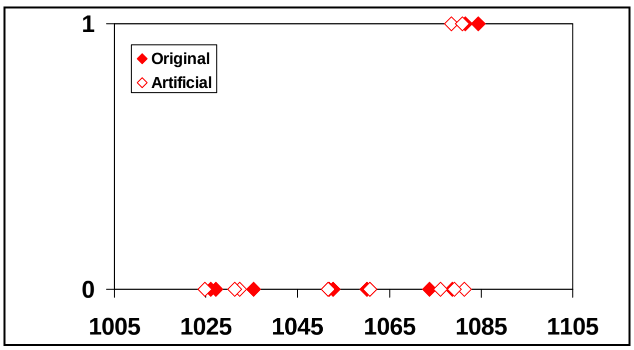

Figure 3. Augmented data set from quantal response study with no zone of mixed results.

The DiDonato-Jarnagin algorithm returns estimates for the mean and standard deviation of 1079.96 and 2.78, respectively. The normal cumulative distribution function for these parameter values (see figure 4) displays the relationship between any RHA penetration prediction and the estimated probability in penetration into the developmental armor. For example, figure 4 also shows that at 1083 units of RHA predicted penetration, the probability of penetration through developmental armor is estimated to be 86.3%.

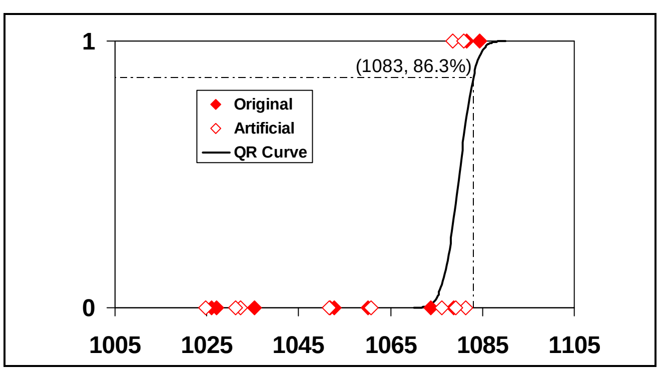

Figure 4. Quantal response curve from augmented data set.

If one repeats this analysis often enough, the resulting empirical distribution can be used to generate a confidence interval for the penetration probability at an RHA penetration of 1083 units. Figure 5 gives a sense of the distribution of these point estimates based on 10,000 repetitions of the analytical process. The point estimates span a considerable range of values. A 90% confidence interval is obtained by choosing the 500th smallest point estimate as the lower limit and the 500th largest point estimate as the upper limit.* The resulting 90% confidence interval is 49.0% and 98.9%. This interval is quite wide, most likely due to the small number of shots (10) that the analysis is based upon.

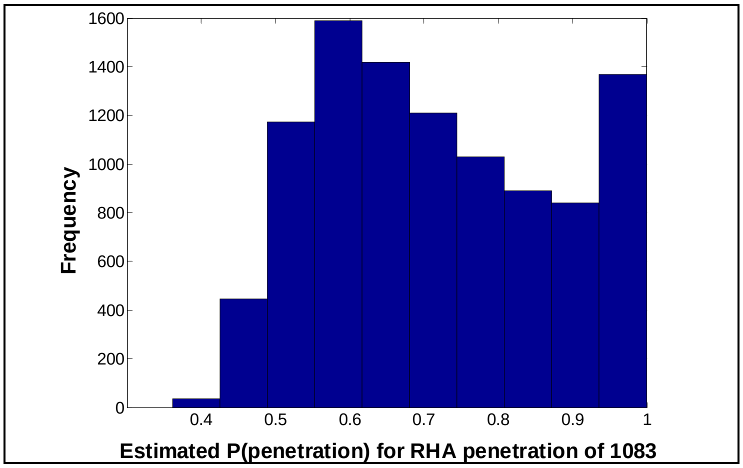

Figure 5. Frequency histogram for 10,000 estimates of developmental armor penetration probability for a predicted RHA penetration of 1083 units.

4. Quantal Response Analysis When the Responses Are Identical (Condition 1 Failure)

Suppose that all of the experiments in a quantal response study result in identical outcomes. Due to limitations on test resources, it may still be desirable to estimate the probability of success at a particular stimulus level. Assume that a study yields successful outcomes at each of the stimulus levels considered. A conservative approach for producing a quantal response curve involves changing the minimum success-producing stimulus level to a failure and then using the augmentation routine detailed in section 2 of this report.† This is likely to induce an upward bias in the estimated mean of the threshold stimuli, meaning that $\hat{\mu}$ is, in general, larger than the true value μ. In turn, this approach will also tend to underestimate success probabilities. For this

* In general, a $100\cdot(1-\alpha)$ percentile two-sided confidence interval is bounded below by the $100\cdot\alpha/2$ percentile and above by the $100\cdot(1-\alpha)/2$ percentile of the distribution of point estimates.

† If all of the observed outcomes are failures, change the maximum stimulus to a success and proceed with the augmentation routine.

reason, it is advised that this method only be used to produce lower confidence bounds for success probabilities at stimulus levels that are greater than (or very close to) the minimum success-producing stimulus level.

5. Example of Augmented Quantal Response Analysis When All Responses Are Successes

Suppose that a quantal response study similar to that described in section 3 yields all penetrations at the following predicted RHA penetrations: 1026.1, 1027.2, 1035.4, 1051.8, 1052.7, 1060.0, 1073.8, 1078.8, 1081.6, and 1084.3. The data is changed so that a failure is assumed to have been observed at the RHA penetration of 1026.1. An analysis is then outlined in sections 2 and 3. Figure 6 shows the alteration to the data.

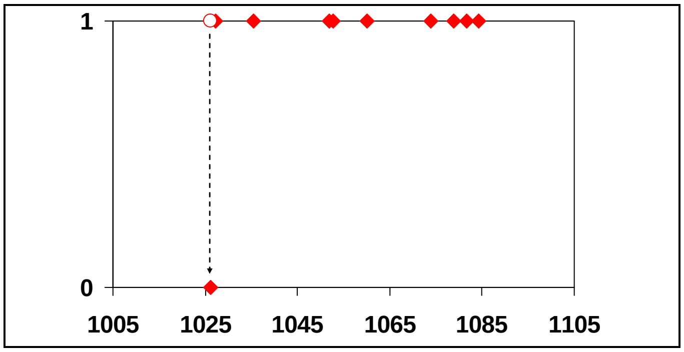

Figure 6. Example of quantal response data in which all outcomes are successes. The outcome with the smallest stimulus level is changed to a failure to facilitate the analysis.

Estimating the probability of penetration for RHA predictions of 1028 and 1080 is required. A large sample of point estimates is then generated for each of these quantities using data augmentation. Figure 7 shows the resulting frequency histograms, giving a sense of the range of possible estimates.

Lower confidence bounds for the developmental armor penetration probabilities at RHA predictions of 1028 and 1080 are 49.5% and 99.97%, respectively. It is clearly seen that there is far greater uncertainty in estimating the success probability at the smaller stimulus level of 1028 units. This is because there is relatively little information about outcomes for stimuli below 1028 units compared to the amount of information known about outcomes below 1080 units.

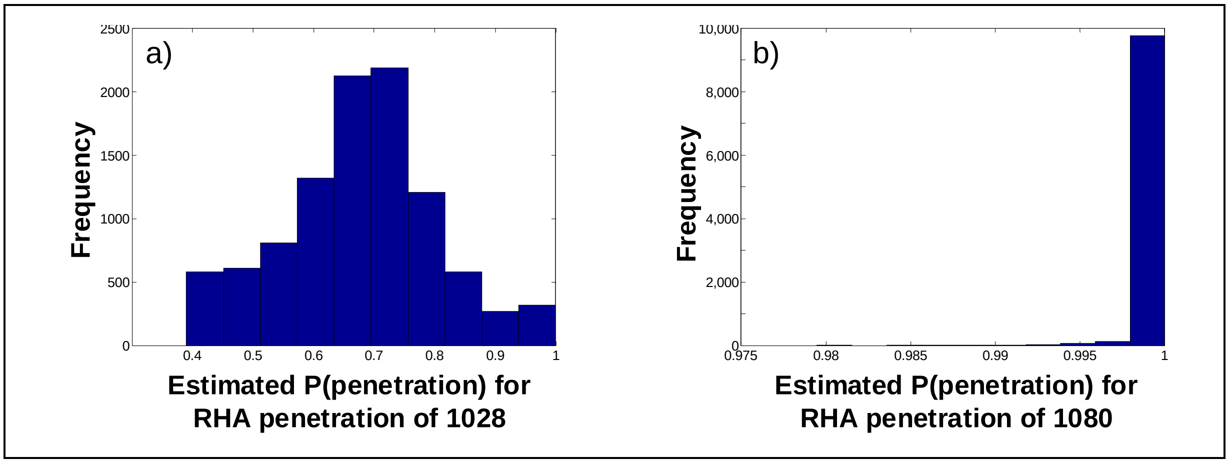

Figure 7. Frequency histograms for 10,000 estimates of developmental armor penetration probability for predicted RHA penetrations of (a) 1028 units and (b) 1080 units.

6. Quantal Response Analysis When the Average Success-Producing Stimulus Is Less Than the Average Failure-Producing Stimulus (Condition 1 Failure)

If the average of the success-producing stimuli is less than the average failure-producing stimuli, then the tendency is for the relationship between stimulus level and probability of success to be decreasing. This contradicts the assumption that this relationship should be increasing. To get around this problem, simply reverse the definitions of success and failure in the study. Under the new definition, proceed with the customary quantal response analysis or the augmentation approach outlined heretofore. Of course, it is necessary to convert the results back into the language and values of the original definition of a successful outcome.

Because this is a trivial case, no example is given. However, it is worth reiterating that the RHA-prediction examples described in sections 3 and 5 were motivated by a survivability-oriented study in which success was originally defined to occur if the armor defeated the projectile.

7. Summary

In this report, the statistical concept of data augmentation has been applied to quantal response analysis when the conditions guaranteeing convergence of the DiDonato-Jarnagin algorithm are not satisfied. This approach was primarily motivated by a need to perform the analysis when a zone of mixed results did not exist. Combining the original data with one or more artificial data sets that have been altered by a random noise component added to the stimulus levels has been postulated. The augmented data set does have a zone of mixed results, allowing one to proceed with the usual maximum-likelihood method using the DiDonato-Jarnagin algorithm.

This novel approach leads to many questions and opportunities for future research. Topics that should be explored include unbiasedness of the estimates, the effects of sample size and choice of standard deviation, and other strategies for generating the artificial data.

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